Upper Bound Calculations on Capacitance of Microstrip Line Using Variational Method and Spectral Domain Approach

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Abstract—**In this paper, the authors have employed an analytical approach based on the Fourier transformation and variational techniques in terms of the surface potential of the dielectric sheet to find the upper bounds of the microstrip line capacitance. It is hoped that our work will complement that of Yamashita *et al.* [5], who calculated the lower bounds dealing with the charge density on the surface of the conductor strip, in estimating the margins of error in calculation.**

I. Introduction

ALTHOUGH microwave circuits were based on the waveguide system until the mid 1960's, the strip line system is now finding extensive application because of its light weight and handiness. A variety of strip line structures are in use, especially the microstrip line shown in Fig. 1(a).

An exact analysis based on a hybrid mode is desirable in determining the parameters of the microstrip line in the frequencies over around 10 GHz. However, in the frequency ranges under X band, a quasi-TEM wave approximation is useful enough. By means of quasi-TEM assumptions, the line capacitance values can be employed in calculating the characteristic impedance and the guide wavelength, so that it is necessary only to solve the two-dimensional Laplace equation over the cross section of the line.

Several solutions for the two-dimensional boundary value problem involving two different media are already known, for example, the modified conformal mapping method [1], the integral equation method [2], [3], the relaxation method [4], and the variational method [5]. Recently, Wheeler [6] has given, with the extensive tables, the simple explicit formulae for the characteristic impedance and the guide wavelength as well as the attenuation of the microstrip line.

Among them, the variational method provides not only a highly precise calculation but also the upper and the lower bounds on true values so that the margins of error in the calculation can be estimated.

An analytical approach based on the Fourier transform and variational techniques was initiated by Yamashita *et al.* [5] in 1968. In their paper, they calculated the lower bounds of the line capacitance dealing with the charge density on the surface of the conductor strip.

Manuscript received August 1, 1977; revised January 27, 1978.
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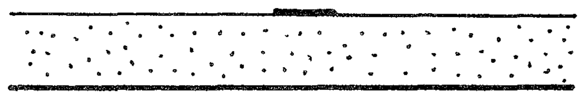
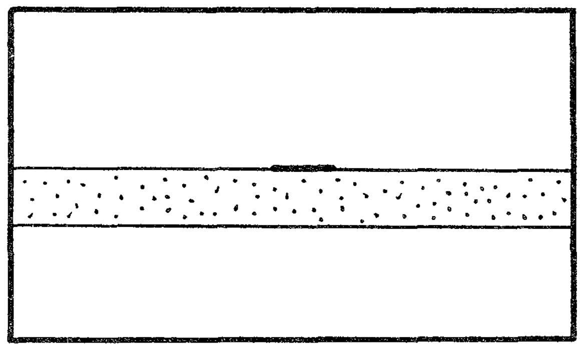


(a)



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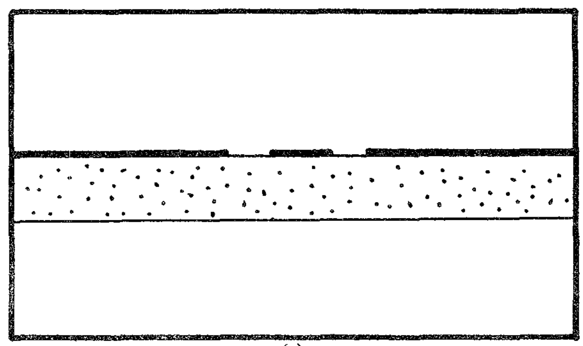

(c)

Fig. 1. Various strip lines.

Strictly speaking, not only the lower bounds but also the upper bounds have to be examined to obtain an approximate value through the variational method. This is because a trial function cannot always be reliable. To contain the margins of error within certain range, both bounds are necessary.

In this paper, an analytical approach based on the Fourier transform and variational techniques to find the upper bounds of the microstrip line capacitance will be discussed in terms of the surface potential of the dielectric sheet.

II. Analysis

It is profitable to use the Fourier transform in the analysis of open structure transmission line shown in Fig. 2.

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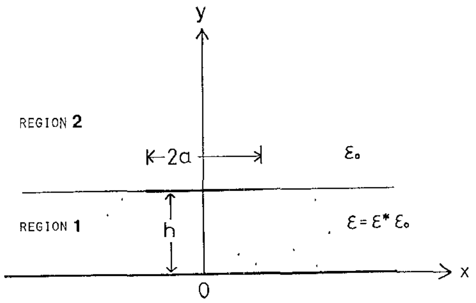

Fig. 2. Line structure.

At first, the Fourier transform is applied to all field quantities, i.e., static potential distributions $\phi_i(x,y)$ ($i=1,2$) in the dielectric region as well as in the air region, and $V(x)$ on the interface, respectively.

$$\tilde{\phi}_i(\beta,y)=\int_{-\infty}^{\infty}\phi_i(x,y)e^{j\beta x}\,dx \tag{1}$$

$$\tilde{V}(\beta)=\int_{-\infty}^{\infty}V(x)e^{j\beta x}\,dx. \tag{2}$$

The static potential distributions $\phi_i(x,y)$ ($i=1,2$) satisfy the Laplace equation in the individual regions:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi_i(x,y)=0. \tag{3}$$

Therefore, the transformed potential distributions $\tilde{\phi}_i(\beta,y)$ ($i=1,2$) satisfy the following ordinary differential equation:

$$\left(-\beta^2+\frac{d^2}{dy^2}\right)\tilde{\phi}_i(\beta,y)=0. \tag{4}$$

Considering the boundary conditions at both the ground plane and the infinity as well as the continuity condition on the interface (at $y=h$), $\tilde{\phi}_i(\beta,y)$ can be expressed in terms of $\tilde{V}(\beta)$ with the following equations:

$$\tilde{\phi}_1(\beta,y)=\tilde{V}(\beta)\cdot\frac{\sinh|\beta|y}{\sinh|\beta|h} \tag{5a}$$

$$\tilde{\phi}_2(\beta,y)=\tilde{V}(\beta)e^{|\beta|(h-y)}. \tag{5b}$$

In the above analysis, we have solved the Dirichlet type boundary value problem in the Fourier transformed domain (spectral domain), but if we were to solve this problem directly, the relation between $V(x)$ and $\phi_i(x,y)$ would be a very complicated form that contains a convolution integral. This is one of the advantages of the Fourier transform method (spectral domain approach) over the conventional method.

Next, we must calculate the total electric energy stored in this system. Applying the vector identity,

$$\int\int(\nabla_t\phi)^2\,ds=\oint\phi\frac{\partial\phi}{\partial n}\,dl-\int\int\phi\nabla_t^2\phi\,ds \tag{6}$$

and Parseval's relation [7],

$$\int_{-\infty}^{\infty}\phi(x)\psi(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\phi}(\beta)\overline{\tilde{\psi}(\beta)}\,d\beta \tag{7}$$

the energy in the dielectric region is expressed also in terms of $\tilde{V}(\beta)$. Thus:

$$\begin{aligned}We_1&=\frac{\epsilon}{2}\cdot\int_0^h\int_{-\infty}^{\infty}(\nabla_t\phi_1)^2\,dx\,dy=\frac{\epsilon}{2}\int_{-\infty}^{\infty}\phi_1\frac{\partial\phi_1}{\partial y}\bigg|_{y=h}dx\\&=\frac{\epsilon}{2}\frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\tilde{\phi}_1(\beta,y)\cdot\overline{\frac{d\tilde{\phi}}{dy}(\beta,y)}\bigg|_{y=h}d\beta\\&=\frac{\epsilon_0}{2}\frac{\epsilon^*}{2\pi}\int_{-\infty}^{\infty}|\tilde{V}(\beta)|^2|\beta|\cdot\coth|\beta|h\,d\beta\end{aligned} \tag{8}$$

where ϵ^* designates a dielectric constant of the substrate. Similarly, the energy stored in the air region is given by (9):

$$\begin{aligned}We_2&=\frac{\epsilon_0}{2}\int_n^{\infty}\int_{-\infty}^{\infty}(\nabla_t\phi_2)^2\,dx\,dy\\&=-\frac{\epsilon_0}{2}\int_{-\infty}^{\infty}\phi_2(x,y)\frac{\partial}{\partial y}\phi_2(x,y)|_{y=h}dx\\&=\frac{\epsilon_0}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{V}(\beta)|^2\cdot|\beta|\,d\beta.\end{aligned} \tag{9}$$

Accordingly, the total energy can be expressed by $\tilde{V}(\beta)$.

$$\begin{aligned}We&=We_1+We_2\\&=\frac{\epsilon_0}{4\pi}\int_{-\infty}^{\infty}|\tilde{V}(\beta)|^2|\beta|\cdot(1+\epsilon^*\coth|\beta|h)\,d\beta.\end{aligned} \tag{10}$$

On the other hand, the line capacitance C can be evaluated by the total energy We.

$$C=\frac{2We}{V^2} \tag{11}$$

where V means the potential at the upper conductor. Therefore, the variational expression for the line capacitance can be obtained by substituting (10) into (11).

$$C=\frac{\epsilon_0}{2\pi}\frac{1}{V^2}\int_{-\infty}^{\infty}|\tilde{V}(\beta)|^2|\beta|\cdot(1+\epsilon^*\coth(\beta)h)\,d\beta. \tag{12}$$

Equation (12) provides the upper bounds on the line capacitance, for the true static potential function $\phi(x,y)$ can give the absolute minimum energy We [8].

If we must find the charge distribution and/or the potential distribution in the original domain, we shall have to deal with the task of the inverse transformation which is very difficult in general, but only the values of the line capacitance is needed here. This is also one of the merits of the Fourier transform method.

Once the line capacitance C and C_0 are evaluated for the loaded and unloaded dielectric sheet, respectively, the characteristic impedance Z and the guide wavelength λ can be obtained as follows:

$$Z=Z_0\sqrt{\frac{C_0}{C}} \tag{13}$$

$$\lambda=\lambda_0\sqrt{\frac{C_0}{C}} \tag{14}$$

where λ_0 means the wavelength in free space and

TABLE I
COMPARISON OF CALCULATED VALUES (= 1)

C/ε_0 \ $2a/h$	Exact Values Eq. (17)	Lower Bounds $\rho(x)=\lvert x\rvert$	Upper Bounds $V(x)=1/x$	Upper Bounds R-R (N=4)
$10^{-2.}$	0.9391	0.9343	1.7533	1.1836
$10^{-1.8}$	1.0095	1.0044	1.7545	1.1955
$10^{-1.6}$	1.0901	1.0856	1.7573	1.2202
$10^{-1.4}$	11848	1.1807	1.7633	1.2621
$10^{-1.2}$	12974	1.2936	1.7760	1.3338
10^{-1}	1.4337	1.4298	1.8020	1.4410

$$Z_0 = \frac{1}{C_0 v} = \frac{120\pi\epsilon_0}{C_0}. \tag{15}$$

III. NUMERICAL RESULTS

By assuming the function of the potential distribution $V(x)$, the line capacitance can be calculated from (12). Even though the formula (12) is a stationary one, we must consider the choice of a trial function. It is advisable to meet the physical boundary conditions as closely as possible, for this will help to obtain a trial field close to the true field. Moreover, we should select such a trial field that has a simple Fourier transform suitable for the numerical calculation.

Needless to say, if we remove the dielectric sheet, the conformal mapping method can be applied to obtain the exact value [9]. So, the exact values compared with the upper bounds and the lower bounds are summarized in Table I. The lower bounds are calculated from the variational expression in which the charge distribution on the conductor $Q(x)$ is employed as a trial function, which is assumed to be of the form $Q(x)=|x|$ [5]. In the calculation of the upper bounds, the trial function $V(x)=1/|x|$ ($|x|>a$) was at first chosen, but a relatively large error occurred (86.7 percent, at $2a/h=10^{-2}$, for example). In the cases where $2a/h \ll 1$ the singularity at the edges of the upper conductor may be very significant. For a "wide" strip the stored energy in the region $|x|>a$ may, of course, diminish relatively and the approximate function $V(x)=1/|x|$ will give better results (2.4 percent at $2a/h=10^{-1.4}$, for example).

In order to obtain the better upper bounds even when $2a/h \ll 1$, it is necessary to take account of the singularity at the edges of the conductor. Therefore, the following potential distribution will be suitable:

$$V(x)=\begin{cases} 1, & |x|<a \\ 1-\dfrac{4}{5}\sqrt{\dfrac{|x|-a}{d}}, & a<|x|<a+d \\ \dfrac{1}{5}\left(\dfrac{d}{|x|-a}\right)^2, & |x|>a+d \end{cases}. \tag{16}$$

Upon employing the trial function of (16), the error can be supressed to 9.6 percent, even at $2a/h=10^{-2}$, but the Fourier transform of (16) becomes

$$\begin{aligned} \tilde{V}(\beta) = \frac{2}{5|\beta|}\cdot\Bigg[& \sin|\beta|(a+d) - \frac{2}{\sqrt{|\beta|d}} \\ & \cdot\{\sin|\beta|a\cdot C(|\beta|d) + \cos|\beta|a\cdot S(|\beta|d) \\ & + d|\beta|\cdot\cos|\beta|(a+d) + (d|\beta|)^2 \\ & \cdot\{\cos|\beta|aSi(|\beta|d) + \sin|\beta|aCi(|\beta|d)\}\Bigg] \end{aligned} \tag{17}$$

where $S(x)$, $C(x)$, $Si(x)$, and $Ci(x)$ are Fresnel sine integral, Fresnel cosine integral, sine integral, and cosine integral, respectively. Since these functions are not elementary ones, a very long computation time is needed (about 5 min per one structure, for example). Hence, this trial function is not practical.

For these reasons $V(x)$ is expanded in terms of a negative power series of x and the expansion coefficients are determined by the Rayleigh–Ritz procedure.

$$V(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_{N+1}}{x^{N+1}}. \tag{18}$$

The condition $V(1)=1$ imposes a constraint among a's, because we can assume $a=1$ without loss of generality.

$$a_1 + a_2 + \cdots + a_{N+1} = 1. \tag{19}$$

Substituting (19) into (18),

$$V(x) = a_1\left(\frac{1}{x} - \frac{1}{x^{N+1}}\right) + \cdots + a_N\left(\frac{1}{x^N} - \frac{1}{x^{N+1}}\right) + \frac{1}{x^{N+1}}. \tag{20}$$

The Fourier transform of (20) is given by (21).

$$\begin{aligned} \tilde{V}(\beta) &= \int_0^\infty V(x) 2\cos\beta x\, dx \\ &= K(\beta) + a_1\psi_1(\beta) + \cdots + a_N\psi_N(\beta) \end{aligned} \tag{21}$$

where

$$K(\beta) = 2\,\frac{\sin\beta}{\beta} + 2\int_1^\infty \frac{\cos\beta x}{x^{N+1}}\, dx \tag{22}$$

and

$$\psi_i(\beta) = 2\int_1^\infty \left(\frac{1}{x^i} - \frac{1}{x^{N+1}}\right)\cos\beta x\, dx. \tag{23}$$

After substituting (21) into (12), C is expressed as the quadratic form of a's, and then a minimization of C with respect to a's, reduces the following matrix equation:

$$[C_{ij}]\cdot[a_j] = -[d_i] \tag{24}$$

where

$$C_{ij} = \int_0^\infty \psi_i(\beta)\psi_j(\beta)h(\beta)\, d\beta \tag{25}$$

$$d_i = \int_0^\infty K(\beta)\psi_i(\beta)h(\beta)\, d\beta \tag{26}$$

TABLE II
CHARACTERISTIC IMPEDANCES

Z/η \ $2a/h$	$\varepsilon^* = 2\ 65$ Upper Bound	Lower Bound	$\varepsilon^* = 8\ 9$ Upper Bound	Lower Bound
10^{-2}	0.7741	0.7031	0.4665	0.4159
$10^{-1.8}$	0.7208	0.6683	0.4330	0.3983
$10^{-1.6}$	0.6402	0.6205	0.3998	0 3784
$10^{-1.4}$	0 6307	0 5931	0.3674	0.3540
$10^{-1.2}$	0.5584	0.5413	0.3337	0.3265
10^{-1}	0.5041	0.4958	0.3007	0.2952

TABLE III
GUIDE WAVELENGTHS

λ/λ_0 \ $2a/h$	$\varepsilon^* = 2.65$ Upper Bound	Lower Bound	$\varepsilon^* = 8\ 9$ Upper Bound	Lower Bound
10^{-2}	0.7269	0.6603	0.4381	0.3906
$10^{-1.8}$	0.7276	0.6749	0 4372	0.4021
$10^{-1.6}$	0.7275	0 6764	0 4359	0.4126
$10^{-1.4}$	0.7261	0.7027	0.4345	0.4194
$10^{-1.2}$	0.7244	0.7023	0.4329	0.4236
10^{-1}	0 7224	0.7108	0.4311	0.4232

and

$$h(\beta)=\beta(1+\epsilon^* \coth \beta h). \tag{27}$$

Thus the line capacitance is given by (28):

$$C=\sum_{i=1}^{N} a_i d_i+\int_0^{\infty} K^2(\beta)\cdot h(\beta) d\beta. \tag{28}$$

Although nonelementary integral functions

$$\int_1^{\infty} \frac{1}{x^i} \cdot \cos \beta x \, dx \tag{29}$$

appear in (22) and (23), these integrals can be written in terms of sine and cosine integrals, using a recursive formula. Therefore, the computation time is almost the same as that for the case where $N=1$, i.e., $V(x)=1/|x|$, requiring, for example, only 30–35 s. The calculated values for $N=4$ are provided in Table I.

Next, the characteristic impedance and the guide wavelength for $\epsilon^*=2.65$ and 8.9 are calculated from (13) and (14), and summarized in Tables II and III, respectively. It should be noted that the calculated values of the characteristic impedance and guide wavelength are lower bound to the true values because of the forms of (13) and (14). In the cases where $2a/h \ll 1$, the absolute error of more than 10 percent occurs. This error is mainly due to the upperbounds values. The error, however, becomes less than 2 percent for $2a/h=10^{-1}$. Needless to say, the accuracy of the calculated values can be improved by increasing N. In most practical cases, a relatively wide strip is used, so that we can conclude that the differences between the lower bounds and the upper bounds are negligibly small for practical purposes.

IV. CONCLUSION

We have analyzed the properties of the microstrip line under the assumption of a quasi-TEM wave valid in the low microwave frequencies. Our analytical approach is based on variational calculation of the line capacitance in the Fourier transformed domain. The potential distribution on the interface between the dielectric sheet and air region has been utilized as a trial function; consequently, the calculated values give the upper bounds. Although results are poor for a very narrow microstrip line, they are good for a moderately wide microstrip line. We have also discussed the margins of error in the variational calculation.

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